Solutions to Equations over Groups acting on Rooted Trees

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Groups acting on rooted trees

Def. The infinite rooted tree of arity n is the set $\{0, \dots, n-1\}^*$ (or n^* for short). The root of the tree is the empty word ϵ.

Def. A tree automorphism of the infinite tree of arity n is a bijection $f : n^* \to n^*$ that fixes the root and preserves incidence.

Def. The **state** of a tree automorphism $f : n^* \to n^*$ at some $v \in n^*$ is the tree automorphism $f@v : n^* \to n^*$ defined by $(f@v)(w) = f(vw)$.

- A tree automorphisms sends subtrees to subtrees.

Def. The **activity** $\mathrm{act}(f)$ of a tree automorphism f is the permutation by which it acts on the first level.

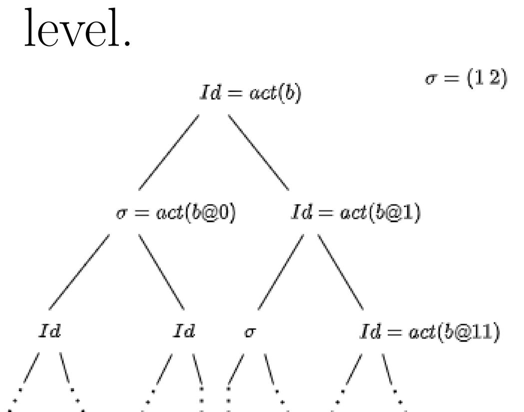

Figure 1: The portrait of a tree automorphism

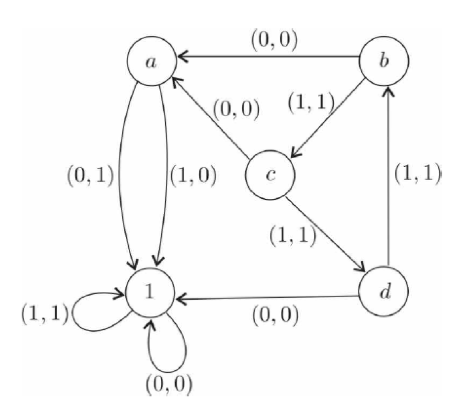

Figure 2: The associated transducer to a tree automorphism

Def. An action of a group G on the infinite rooted tree is a group homomorphism $G \to \mathrm{Aut}(n^*)$.

Def. A group G acting on an infinite tree is **fractal** if the section map $\varphi : G \to \mathrm{Aut}(T^*)$, $g \mapsto (g@0, \dots, g@n-1)$ has image G.

Def. A fractal group G is **contracting** if there is a finite set N (the **nucleus** of G) such that for each f there is only finitely many $w \in n^*$ such that $f@w \notin N$.

Def. A fractal group G is regular branch if it has a finite index subgroup K such that $K^n \leq \psi(K)$.

Tree languages

Def. If T is a (finite or infinite) tree and A a (finite) set, then a map $T \to A$ is a **labelling** of T by the alphabet A.

Def. A **formal language of infinite trees** is a set of labellings over an alphabet A of some fixed infinite tree T.

Def. A **tree automaton** on k-ary trees labelled by A is a tuple (Q, q_0, Δ, F) where
- Q is a finite set of states,
- $q_0 \in Q$ is the initial state,
- $\Delta \subseteq Q \times A \times Q^k$ is the transition relation,
- F is a set of accepting states.

- A tree automaton can perform an acceptance check on a labelled tree: Start at the root of the tree in the initial state q_0. If the root is labelled by $a \in A$, choose q_1, q_2 s.t. $(q_0, a, q_1, q_2) \in \Delta$. Then proceed recursively on the left subtree with q_1 and on the right subtree with q_1. If there are choices of transitions in Δ such that along each path in the tree infinitely many accepting states are reached, the tree is accepted.

Def. A formal language of infinite trees is **regular** if it is accepted by a tree automaton.

- The class of regular tree languages is closed under union, intersection and complement and has decidable emptiness problem [Rabin 1964].

Theorem [Niwinski 1991]

If L is a countable regular language of infinite trees, there is a finite set X of infinite regular trees and a tree automaton $\mathcal{A}$ on the alphabet $A \cup X$ such that

$$L = \mathcal{L}(\mathcal{A}, X) := \mathcal{L}^{fin}(\mathcal{A})_{\to X}$$

Theorem [Bartholdi-Noce 2022]

The set of portraits of a contracting regular branch group is a regular tree language.

Equations over self-similar groups

The goal here is to solve (systems of) equations over groups, aka existentially quantified formulas over the group language. For fractal (self-similar) groups, one can utilize the recursive structure:

- Suppose we are given the equation $aXbY = 1$.
- Solutions for X, Y in a self similar group have form $X = \mathrm{act}(X)(X_0, X_1)$ and $Y = \mathrm{act}(Y)(Y_0, Y_1)$.
- Hence the equation is equivalent to the three conditions $\mathrm{act}(a)\,\mathrm{act}(X)\,\mathrm{act}(b)\,\mathrm{act}(Y) = \mathrm{id}$,
$$a_0 X_i b_j Y_k = 1 \text{ and } a_1 X_{i'} b_{j'} Y_{k'} = 1.$$
- Idea: Repeatedly applying this procedure reduces to a finite system of equations with "small" coefficients.

Theorem [Lysenok et al. 2013]

It is decidable whether a quadratic equation has a solution in the Grigorchuk group.

Theorem [Sidki et al. 2018]

The simultaneous conjugacy problem is solvable in the group of bounded tree automorphisms.

Bottom up algorithms

- Solve the equation over the nucleus of the group - this is a finite amount of computations.
- Perform a reachability analysis on the tree automaton. Can we reach the starting state from the accepting states?
- Use computed solutions of the equations on lower levels of the automaton to build solutions on higher levels.
- For now: Stick to equations with one variable, i.e. $X^k = 1, X^k = a, \dots$

An algorithm for $X^2 = 1$

Input: A finite set $F \subseteq \mathrm{Trees}^2_{S_2}$ for which inverses and being order 2 is known.
Input: A tree automaton (Q, s_0, Δ, F) on $S_2 \cup F$-labelled binary trees.
Output: The set $Tor = \{q \in Q \mid \exists f \in \mathcal{L}(\mathcal{A}_q, F) : f^2 = \mathrm{id}\}$

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Inv ← {(p, q) ∈ F² | p · q = id}
Tor ← {q ∈ F | q² = id}
while size of Inv or Tor increases do
    for all rule pairs (q, a, q1, q2), (q', a', q1', q2') ∈ Δ do
        if    a = a' = 1 ∧ (q1, q1') ∈ Inv ∧ (q2, q2') ∈ Inv
          or a = a' = σ ∧ (q1, q2') ∈ Inv ∧ (q2, q1') ∈ Inv then
            Inv ← Inv ∪ {q, q'}
        end if
    end for
    for all rules (q, a, q1, q2) ∈ Δ do
        if    a = id ∧ q1 ∈ Tor ∧ q2 ∈ Tor
          or a = σ ∧ (q1, q2) ∈ Inv then
            Tor ← Tor ∪ {q}
        end if
    end for
end while
return Tor
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Results

Theorem

It is decidable whether a contracting regular branch group is torsion-free.

Future Goals

- Can this method be generalized to arbitrary equations?
- Is the theory of contracting regular branch groups decidable?

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