Capturing structure of infinite group presentations

Using formal language theory and automata

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Group presentations

Denote by F_S the free group with generating set S and by $\langle\langle R\rangle\rangle^G$ the normal closure of R in G.

Definition

A **group presentation** is a set S of generators together with a set of words R over $S \cup S^{-1}$.
It defines a group

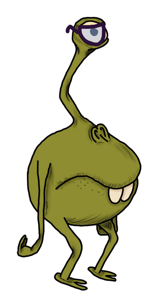
$$\langle S|R\rangle = F_S/\langle\langle R\rangle\rangle^{F_S}$$

Examples: $\langle a, b|\ \rangle \cong F_2$ $\quad \langle a, b|aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2$ $\quad \langle x_1, x_2, \ldots |x_n^n = x_{n-1}, n \in \mathbb{N}\rangle \cong \mathbb{Q}$

Formal Language Theory

Consider a finite set Σ which we call the *alphabet* and its elements the *letters*.

- A *formal language* is any set $L \subseteq \Sigma^* = \{$all finite words over $\Sigma\}$.
- Examples over the alphabet $\Sigma = \{a, b\}$: $\{a, aa, b\}$, $\{a^{2n} \mid n \in \mathbb{N}\}$, $\{a^n b^n \mid n \in \mathbb{N}\}$.
- A *language class* is a collection of languages, independent of the alphabet.
- Examples: The class FIN of finite languages, the class REG of regular languages, the class CF of context-free languages, the class REC of recursive languages.



Definition

Consider a language class LAN. A group G has a

LAN-*presentation*

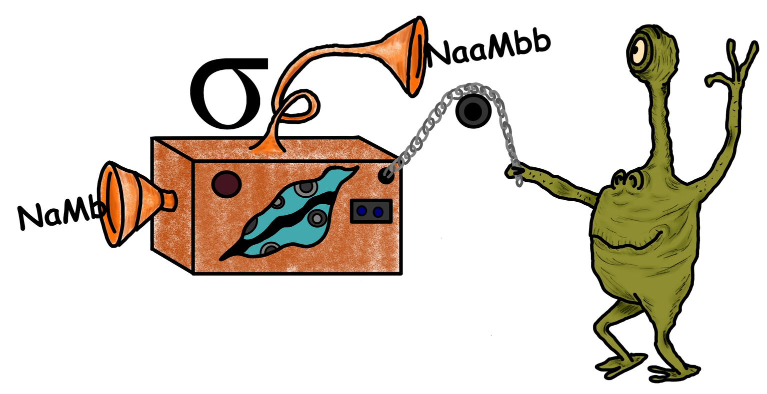
if it has a presentation $G = \langle S \mid R\rangle$ such that R, considered as a language over the alphabet $S \cup S^{-1}$, is in the class LAN.

The class of groups that have a LAN-presentation is denoted by $\Gamma(\mathrm{LAN})$.

EDT0L languages

- An *EDT0L system* over some alphabet Σ consists of a finite set T ("non-terminals") and finitely many *rewriting rules* $\Sigma \to \Sigma^*$ and a *start word* $w_0 \in (\Sigma \cup T)^*$.
- The *language* L of an EDT0L system is produced as follows:
  - Beginning with the start word, pick any rewriting rule and apply it *to the entire word*.
  - If the resulting word only contains letters in Σ, add it to L.
  - In any case, take the resulting word and again pick a rewriting rule and continue.
  - Do this in all possible combinations of rewriting rules.
- In formulas: $\mathcal{L}(T, \Phi, w_0) = \Sigma^* \cap \bigcup_{n\in\mathbb{N}} \Phi^n(w_0)$.

Example:
Starting word: $N_a N_b N_c$
Rewriting rule σ: $\sigma(a) = a, \sigma(b) = b, \sigma(c) = c$
$\sigma(N_a) = N_a a, \sigma(N_b) = N_b b, \sigma(N_c) = N_c c$
Rewriting rule τ: $\tau(a) = a, \tau(b) = b, \tau(c) = c$
$\tau(N_a) = \tau(N_b) = \tau(N_c) = \epsilon$
Language: $\{a^n b^n c^n \mid n \in \mathbb{N}\}$



Theorem [Lysenok'85]

The **Grigorchuk group** has a EDT0L (DF0L) presentation $\mathcal{G} = \langle a, c, d|R_\mathcal{G}\rangle$ where $R_\mathcal{G}$ is given by the following EDT0L system:
Starting words: $\{a^2, [d, d^a], [d^{ac}, d^{aca}]\}$
Rewriting rule σ: $\sigma(a) = aca, \sigma(c) = cd, \sigma(d) = c$

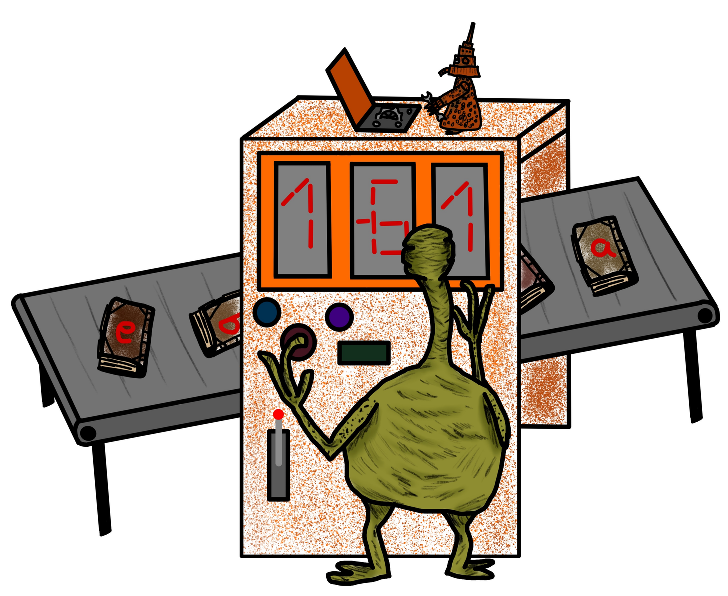
Why are EDT0L presentations interesting? Let $\mathcal{C}$ be the set of abelian-by-nilpotent groups, or that of groups that are virtual direct products of finitely many hyperbolic groups.

Theorem [Bartholdi, P., Rauzy]

There is an algorithm that takes as input an EDT0L presentation of a group G and a finite presentation for a group H in the class $\mathcal{C}$, that decides whether or not H is a quotient of G.

m-counter languages

- For $m \in \mathbb{N}$, an *m-counter automaton* of the alphabet Σ consists of
  - A finite set Q of *states*,
  - A *transition relation* $\delta \subseteq Q \times \Sigma \times \mathbb{Z}^m \times Q$.
  - An *initial state* $q_0 \in Q$ and a set $F \subseteq Q$ of *accepting states*.
- The *language* of an m-counter automaton is produced as follows:
  - Take an input word $w \in \Sigma^*$. Start at the starting state q_0 and follow the transition relation: $(q_0, w_0, z_1, q_1)\delta \to (q_1, w_1, z_2, q_2) \to \ldots$.
  - The *accumulated memory value* is the sum $z_1 + z_2 + \ldots$.
  - If you can reach an accepting state with accumulated memory value equal to 0, then add w to the language.



Example languages:

$\{a^n b^n c^n d^n \mid n \in \mathbb{N}\}$

$\{w \in \Sigma^* \mid |w|_a = |w|_b\}$

$\{a^n b^m c^{n+m} \mid n, m \in \mathbb{N}\}$

$\mathrm{WP}(\mathbb{Z}^n)$

The *lamplighter group* $L_2 = \mathbb{Z}_2 \wr \mathbb{Z} = \langle a, t \mid a^2, [t^n a t^{-n}, a]_{n\in\mathbb{Z}}\rangle$ has a 3-counter presentation.

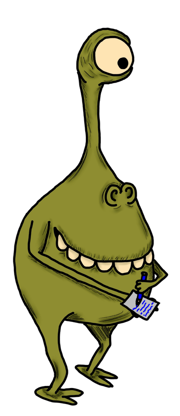
Theorem [P., Vandeputte]

Consider groups G and H.

$\Rightarrow$ If G and H have m-counter presentations, then $G \wr H$ has a $3m$-counter presentation.

$\Leftarrow$ If $G \wr H$ has an m-counter presentation, then so do G and H. If H is abelian, then it has rank at most $m + 1$.

From Language Classes...



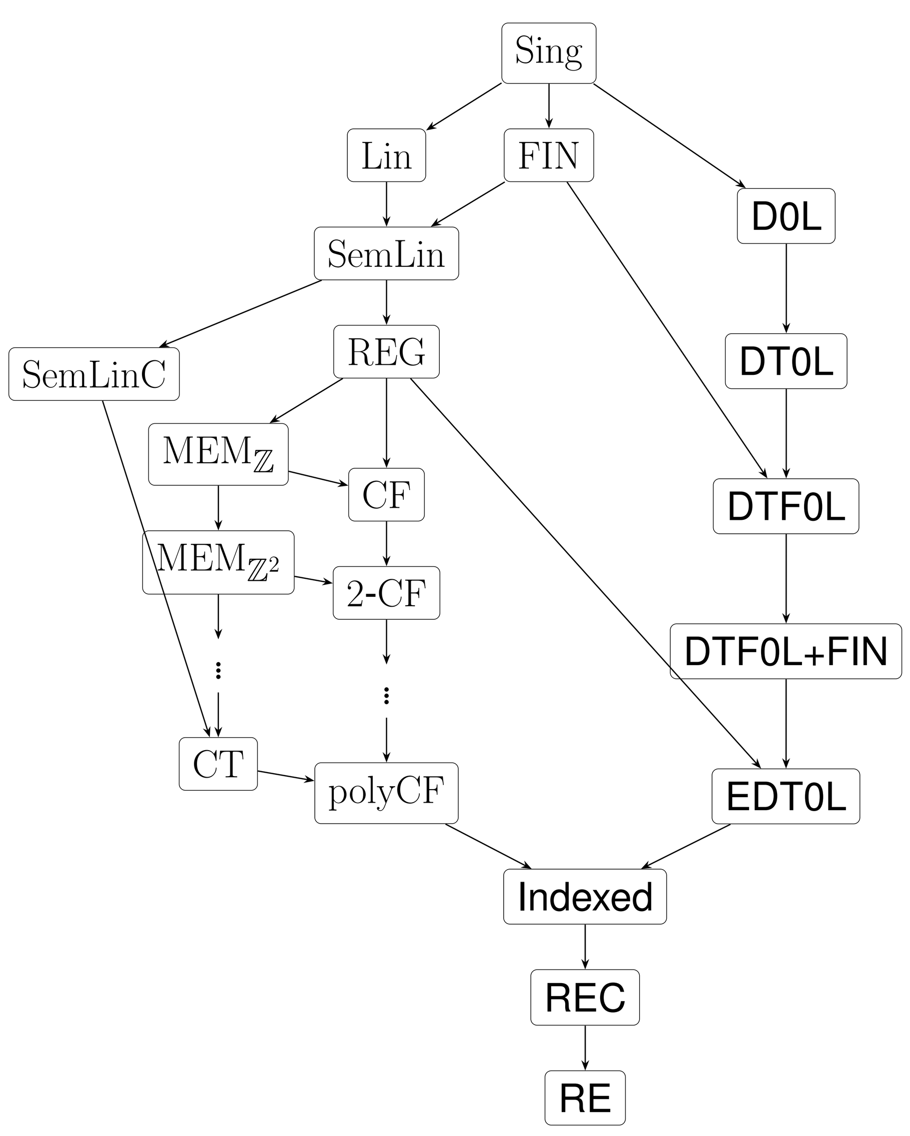


Figure 1: Hierarchy of language classes.

...to Group Classes.

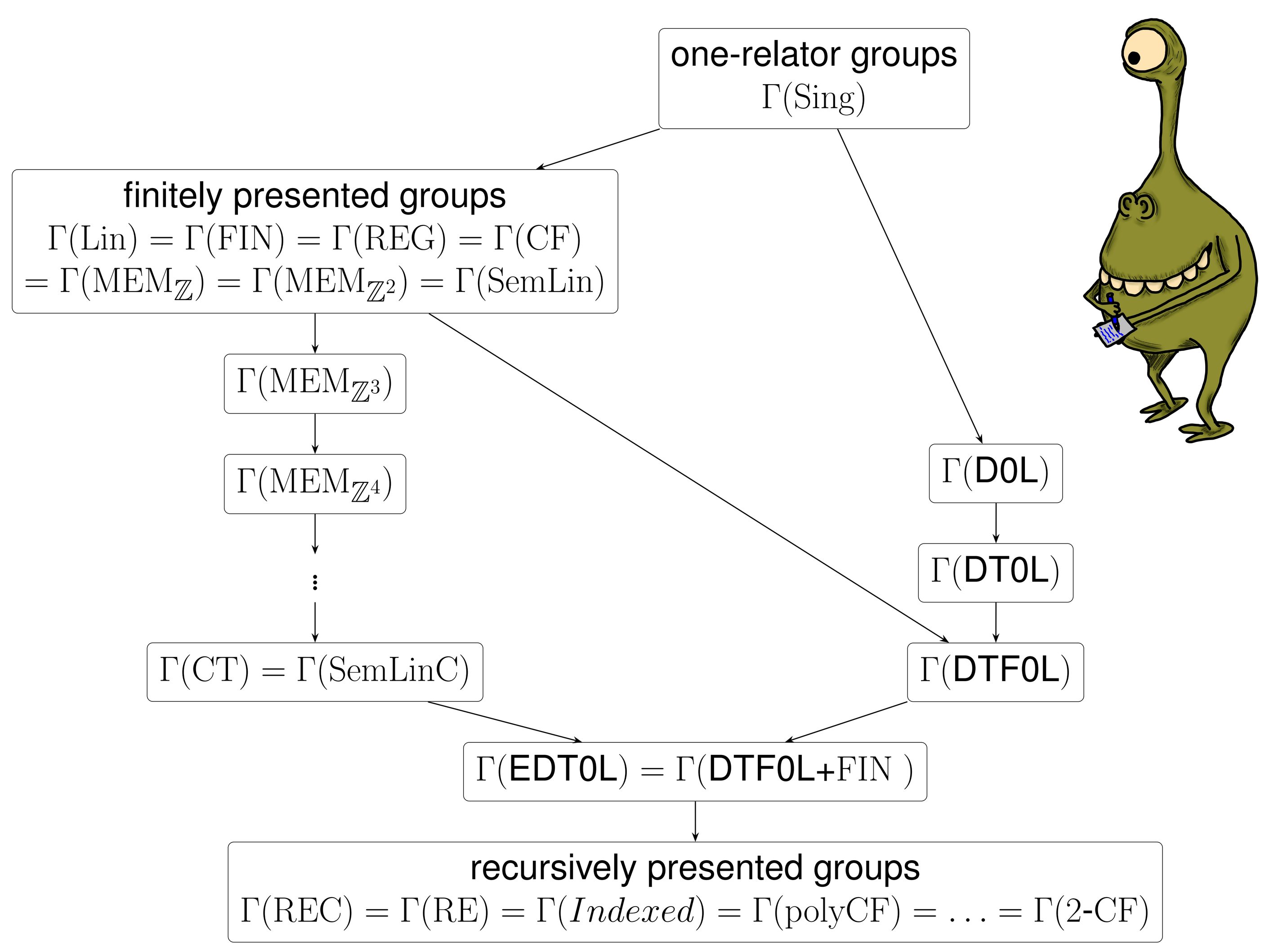


Figure 2: Hierarchy of group classes.

Above is an overview over the relations between different language classes and the resulting group classes. Arrows symbolize strict containment. Notice the various collapses: Many language classes actually produce the same groups. The m-counter hierarchy does not collapse, but rather allows to order the EDT0L-presented groups.

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