EDT0L presentations of groups

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Presentations of Groups

- Denote by F_S the free group with generating set S and by $\langle\langle R\rangle\rangle^G$ the normal closure of R in G.

Def. A **group presentation** is a set S of generators together with a set of words R over $S \cup S^{-1}$. It defines a group

$$\langle S|R\rangle = F_S/\langle\langle R\rangle\rangle^{F_S}$$

Examples

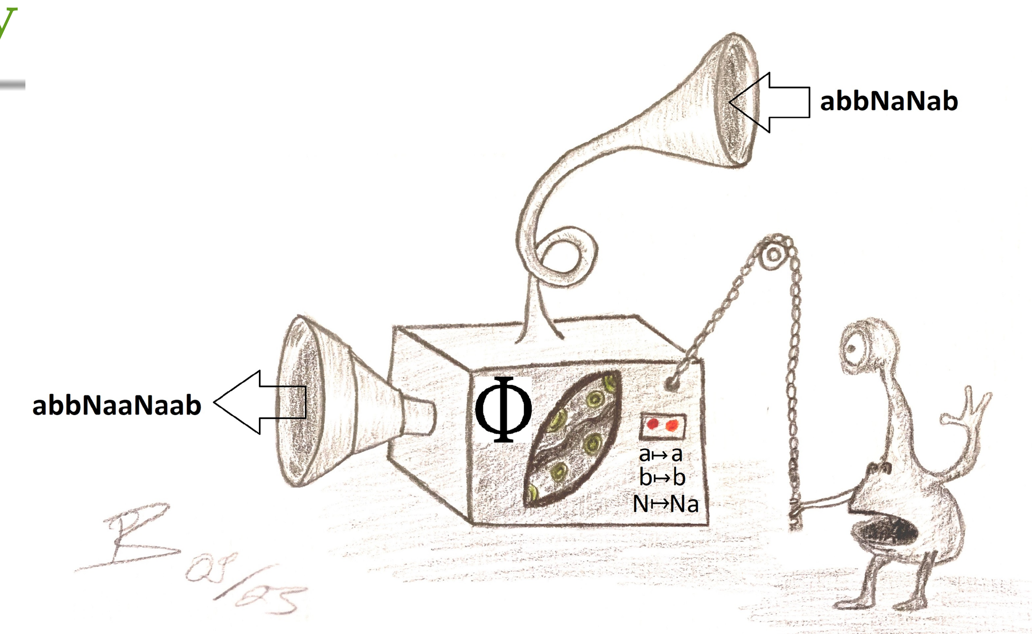
$\langle a, b|\ \rangle \cong F_2$ $\quad$ $\langle a, b|aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2$ $\quad$ $\langle x_1, x_2, \ldots |x_n^n = x_{n-1}, n \in \mathbb{N}\rangle \cong \mathbb{Q}$

Definition

A group G has an **EDT0L presentation** if it has a presentation $G = \langle S \mid R\rangle$ such that R is an EDT0L language over the alphabet $S \cup S^{-1}$.

Language Theory

- Fix a finite set A which we call the **alphabet**. A **formal language** over A is a subset $L \subseteq A^*$ of the finite words over A.
- One way of producing a formal language from a finite description is through variations of so called L-Systems. They are based on **parallel rewriting rules**, i.e. maps $A \to A^*$ which map every letter to a word over the alphabet.



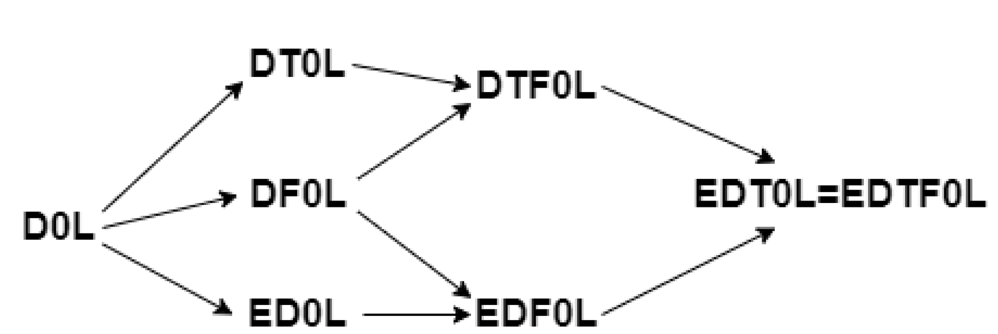
- An L-system consists of a starting word and a rewriting rule. The language of the system is generated by repeatedly applying the rewriting rule to the starting word. On top of this, one imposes additional filter mechanisms, which define the different types of L-systems. A few of them are listed below.

D	"Deterministic" - Generally, rewriting rules are allowed to map one letter to more than one word. In that case, all combinations of choices will produce a new set of words from a starting word. If the system is deterministic, the rewriting rule is 1:1.
T	"Table" - Multiple rewriting rules allowed, one is chosen for each step in the production process.
F	"Finite start" - Instead of only one starting word, multiple can be specified.
E	"Extended" - The alphabet is extended by a finite number of special symbols (non-terminals). Words containing non-terminals that are produced in the process will not belong to the resulting language, but are handled in the same way for everything else.

Examples

D0L system:
$A = \{a, b\}$
$\omega_0 = ab$
$\Phi : a \mapsto aa, b \mapsto bb$
$\Rightarrow L = \{a^{2^n}b^{2^n} \mid n \in \mathbb{N}\}$

EDT0L system:
$A = \{a, b\} \cup \{M_a, M_b, M_c\}$
$\omega_0 = M_aM_bM_c$
$\Phi_1(M_x) = M_xx,\ \Phi_2(M_x) = x$
$\Rightarrow L = \{a^nb^nc^n \mid n \in \mathbb{N}\}$



The containment relations between some classes of L-languages.
An arrow means proper containment.

Some examples of (E)D(T)0L presentations:

Free abelian group on n generators
D0L presentation
$\langle x_1, \ldots, x_n \mid R_{fa}\rangle$
start word $[x_1, x_2]$
rewriting rule
$\phi(x_i) = x_{i+1}$
$\phi(x_n) = x_1$

[Lysionok'85]
Grigorchuk group:
DF0L presentation
$\mathcal{G} = \langle a, c, d|R_{\mathcal{G}}\rangle$
start words
$\{a^2, [d, d^a], [d^{ac}, d^{aca}]\}$
rewriting rule
$\sigma(a) = aca$
$\sigma(c) = cd$
$\sigma(d) = c$

The Grigorchuk group is a special case of:

Theorem [Bartholdi 2002]

A finitely generated contracting self-similar regular branch group has an EDT0L presentation.

Use of EDT0L presentations

- To study the structure of a group, it is helpful to understand its possible quotients.
- As arbitrary quotients can be quite complex, and in particular evade finite representability which is crucial for algorithmic purposes, it is natural to focus on subclasses of quotients.
- One is the class of maximal nilpotent quotients, i.e. quotients of the group G by members of its lower central series $\gamma_i(G) = [\gamma_{i-1}(G), G]$, $\gamma_0(G) = G$. As finitely generated nilpotent groups, theses are finitely presented.

Theorem [Barth., Eick, Hartung]

There is an algorithm that, given an EDT0L presentation for a group G and $i \in \mathbb{N}$, computes a finite presentation for the group $G/\gamma_i(G)$.

Algorithm

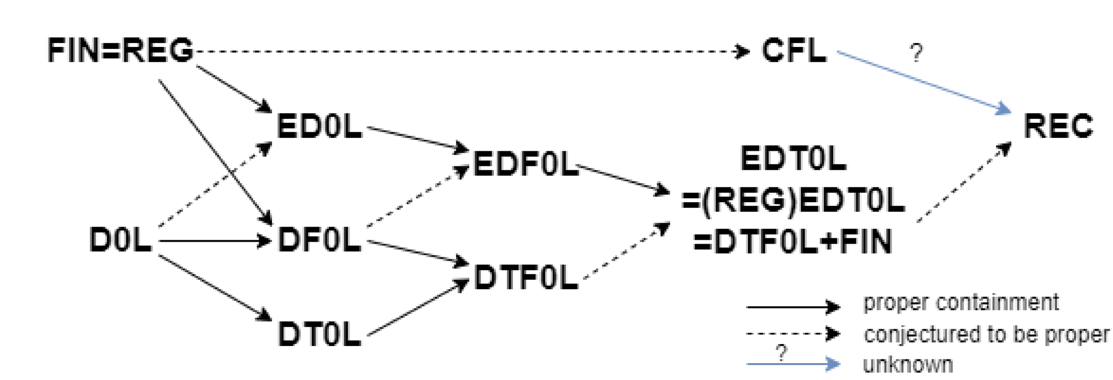
1. Compute $\cup_{i=0}^n \Phi^i(w_0)$ and $\Phi^{i+1}(w_0)$
2. Check whether the words $\Phi^{i+1}(w_0)$ are trivial in $\langle S \mid N_k(S), \cup_{i=0}^n \Phi^i(w_0)\rangle$.
3. If no, repeat for $i + 1$. If yes, return presentation $\langle T \mid N_k(T), f(\cup_{i=0}^n \Phi^i(w_0))\rangle$.

w_0: start word $\quad$ $f : S \to T$: morphism
Φ: rewriting rules $\quad$ N_k: k-iterated commutators

In fact, all the results on the left have been formulated in their time without the use of EDT0L language theory, but in the similar concept of L-presentations. We could however prove equivalence with EDT0L presentations which have to our knowledge not been studied before. This allowed for a clarification of proofs in some cases and we believe that EDT0L presentations are in fact the more natural approach to theses kinds of groups, as well as allowing a study of their close relatives like DTF0L presented groups etc. We also managed to prove that finite quotients of EDT0L presented groups are computable, which was not known before.

Classes of presentations

We also studied the relatives of EDT0L presented groups, i.e. groups with other types of L-systems defining their presentations. Below are some containment relations between classes of groups with presentations in different language classes.



Note that containment relations differ from the ones inferred from the language classes themselves.

Future Endeavours

- Fill in the gaps in the above graphic on class containment in group presentations.Find example to properly distinguish EDT0L-presented groups from groups with a recursive (aka computable) presentation.
- Explore other ways algorithms can exploit the iterative generation process of EDT0L presentations.

References

[1] Bartholdi, Eick, and Hartung. A nilpotent quotient algorithm for certain infinitely presented groups and its applications. *International Journal of Algebra and Computation 18*, 08 (12 2008).

[2] Bartholdi, L. Endomorphic presentations of branch groups. *Journal of Algebra 268*, 2 (10 2003), 419–443.

[3] Lysenok, I. G. A system of defining relations for a grigorchuk group. *Mathematical Notes of the Academy of Sciences of the USSR 38*, 4 (10 1985), 784–792.

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